Finite Quantized-Output Feedback Tracking Control of Possibly Non-Minimum Phase Linear Systems

Jian Guo, Yanjun Zhang, *Member, IEEE*, Ji-Feng Zhang, *Fellow, IEEE*, and Xiao-Kang Liu, *Member, IEEE*

***Abstract*—This paper proposes a finite quantized-output feedback tracking control method for a general class of continuous-time linear time-invariant systems. Firstly, an analytical pole placement-based control law is constructed using a finite quantized-output signal and an external reference output signal. Then, it is proven that the proposed control law can ensure all closed-loop signals are bounded, and the output tracking error converges to a residual set of the origin exponentially. Moreover, we show that the residual set can be made arbitrarily small if the magnitude of the quantizer satisfies a specified condition. This method combines the advantages of the classical pole placement control technique and the quantized control technique. Compared with the existing tracking control methods, it not only reduces the feedback information requirement, but also has full capability to deal with unstable poles and zeros in the controlled plant for output tracking. A numerical example is presented to verify the validity and new features of the proposed method.**

***Index Terms*—Continuous-time, finite time, output tracking, pole placement control, quantized-output feedback.**

I. Introduction

THE CONTROL systems subject to quantization and/or saturation constraints have attracted a lot of attention in the control community. On the one hand, measurement errors and saturation constraints often exist in real control systems due to sensor accuracy and magnitude limitations. On the other hand, quantized feedback control has specific advantages over precise feedback control [1]–[4]. Classical state and output feedback control methods generally cannot be directly applied to control systems subject to quantization and saturation measurements, especially in the case of finite quantization. It is of significant theoretical and practical interest to systematically study quantized control systems.

The quantizers in control systems are generally classified into two types according to whether the quantizer is static or dynamic. A static quantizer is typically easy to use, but it may not be easy to achieve global convergence and often needs infinite quantization levels. Because of this, a dynamic quantizer is first developed in [5]. Compared with the static quantizer, the dynamic quantizer generally has a tunable parameter called the "sensitivity". Dynamic quantization also has promising applications in practice, such as vision-based control, about which a comprehensive description can be found in [5]. Up to the present, we have witnessed tremendous developments in quantized feedback control theory and applications [6]–[16].

Although great progress has been made in quantized control theory and applications, some open problems still need to be addressed. In particular, [5] extensively studied the stabilizing control problems for linear time-invariant systems (LTI) systems. It motivates us to consider whether the stabilizing method in [5] can be extended to output tracking control of general continuous-time LTI systems, especially for the non-minimum phase case. Recently, we have solved the model reference control (MRC) problem of discrete-time LTI systems by quantized-output feedback in [17]. However, the method in [17] is only applicable to control minimum-phase systems. Moreover, the proposed method in [17] is not effective for the controlled plant addressed in this paper due to the essential differences between the stability characterizations of the continuous-time and discrete-time systems. Therefore, to efficiently control a general class of continuous-time LTI systems covering both minimum-phase and non-minimum phase, we plan to use the well-known pole placement control (PPC) method. For a detailed introduction to PPC, the readers may be referred to [18]–[20]. This paper will for the first time address output tracking control of a general class of continuous-time LTI systems covering minimum-phase and non-minimum phase by using finite quantized-output feedback. By the way, the quantization based control methods have been applied for solving some vision-based control problems in [21] and some

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mobile vehicles control problems [22]. The contributions of this paper are as follows.

- A finite quantized-output feedback method is proposed for a general class of continuous-time LTI systems. We show that a finite quantized-output feedback version of the classical PPC law guarantees closed-loop stability and bounded output tracking for the above systems.
- Compared with existing literature, the proposed control method has its distinctive characteristics: the proposed control law is constructed only by using the external reference input and the finite-and-quantized output; the controlled plant is allowed to have unstable poles and zeros, that is, the proposed method is effective for both the minimum-phase and non-minimum phase systems; and the tracking result is global in the sense that the proposed control law is independent of the initial conditions.

The rest of the paper is organized as follows. Section II presents the problem statement. Section III designs the quantized-output feedback PPC law and gives the stability and output tracking analysis. Section IV illustrates a simulation example to verify the effectiveness of the proposed control method. Section V gives some concluding remarks.

Notation: In this paper, we use $\mathbb{R}^+$ and $\mathbb{Z}^+$ to denote the sets of positive real numbers and positive integers, respectively. By $\mathcal{L}$ and $\mathcal{L}^{-1}$, we denote the Laplace transform and the inverse Laplace transform. Let s denote the differentiation operator or Laplace transform operator, i.e., $s[x](t) = \dot{x}(t)$ or $\mathcal{L}(x(t))$ with $x(t) \in \mathbb{R}^n$, $t \geq t_0$. By L^∞ and L^1, we denote two signal spaces defined as $L^\infty = \{x(t) : \|x(\cdot)\|_\infty < \infty\}$ and $L^1 = \{x(t) : \|x(\cdot)\|_1 < \infty\}$ with $\|x(\cdot)\|_\infty = \sup_{t\geq t_0} \|x(t)\|_\infty$ and $\|x(\cdot)\|_1 = \int_{t_0}^{\infty} \|x(t)\|_1 dt$, respectively. For any scalar transfer function $G(s)$, we denote $\|G\|_1 = \int_{t_0}^{\infty} |\mathcal{L}^{-1}(G(s))(t)| dt$ as the L^1 operator norm of $G(s)$. For any $X \in \mathbb{R}$, we let $[X] = \max\{k \in \mathbb{Z} : k < X\}$. In particular, we use the notation: $y(t) = G(s)[u](t)$, to denote the output $y(t)$ of a continuous-time LTI system represented by a transfer function $G(s)$ with input $u(t)$. This notation is simple to combine both time and s-domain signal operations, helpful for control design and analysis, and useful to avoid complex convolution expressions for control system presentation.

II. Problem Statement

In this section, we present the system model to be studied and the problems to be solved.

System model: Consider the following continuous-time single-input and single-output (SISO) LTI system:

$$P(s)[y](t) = Z(s)[u](t), \ \ t \geq t_0, \tag{1}$$

where $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ are the output and input, respectively, t_0 is the initial moment of the system operation, and $P(s)$ and $Z(s)$ are some polynomials with constant coefficients and of degrees n and $n-1$, respectively, i.e.,

$$P(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0,$$
$$Z(s) = z_{n-1}s^{n-1} + z_{n-2}s^{n-2} + \cdots + z_1 s + z_0.$$

For the system (1), we consider the case when $y(t)$ cannot be measured accurately. We can only acquire finite quantized values of $y(t)$ for feedback control. This paper uses $q(y(t), \Delta(t))$ to denote the quantizer, where $\Delta(t)$ is a time-varying signal called the sensitivity.

Quantizer model: The quantizer of this paper is the same as that in [5]. It contains two parameters: a constant parameter $M \in \mathbb{Z}^+$ and a time-varying parameter $\Delta(t) \in \mathbb{R}^+$. The two parameters are called the saturation value and quantization sensitivity, respectively [5]. The quantizer model is

$$q(y(t), \Delta(t)) = \begin{cases} M, & \text{if } y(t) > (M + \frac{1}{2})\Delta(t), \\ \left[\frac{y(t)}{\Delta(t)} + \frac{1}{2}\right], & \text{if } -(M + \frac{1}{2})\Delta(t) < y(t) \leq (M + \frac{1}{2})\Delta(t), \\ -M, & \text{if } y(t) \leq -(M + \frac{1}{2})\Delta(t). \end{cases}$$

Reference output model: The reference output signal, denoted as $y^*(t)$, is bounded and satisfies the condition:

$$Q_m(s)[y^*](t) = 0, \tag{2}$$

where $Q_m(s) = s^{n_q} + q_{n_q-1}s^{n_q-1} + \cdots + q_1 s + q_0$ is a monic polynomial of degree n_q such that the zeros of $Q_m(s)$ are either in the open left-half on the complex plane or on the $j\omega$-axis but not repeated.

Remark 1: In this paper, the reference signal needs to satisfy the condition (2). As we know, under an MRC control framework, the reference signal only needs to be bounded. It is worth noting that the MRC method can only handle the minimum-phase systems, however, the proposed method in this paper not only can handle the minimum-phase systems, but is also effective for control of non-minimum phase systems. The condition (2) on $y^*(t)$ is not restrictive. For many bounded signals, we can choose some appropriate $Q_m(s)$ to satisfy (2). A detailed explanation can be seen in [20].

Control objective: For any given bounded reference signal $y^*(t) \in L^\infty$ satisfying (2), the control objective is to develop a finite quantized-output feedback control law $u(t)$, only using the finite-and-quantized output $q(y(t), \Delta(t))$ and the reference output signal $y^*(t)$, for the system model (1) to ensure that all closed-loop signals are bounded and $y(t) - y^*(t)$ converges to a certain small residual set exponentially.

Assumption: The only assumption needed for control design is as follows.

(A1): $P(s)Q_m(s)$ and $Z(s)$ are coprime.

Assumption (A1) is a standard condition in classical PPC for continuous-time LTI systems [20]. We will show that the quantized-output feedback version of the classical PPC law is valid under Assumption (A1).

III. Quantized-Output Feedback Control Design

This section presents the design details of the finite quantized-output feedback control for the system (1). The whole procedure mainly contains four steps. Firstly, we give a key design equation that will be used to calculate some parameters for the control law design. Then we construct a finite quantized-output feedback control law structure and derive a tracking error equation. Thirdly, we derive some key technical lemmas crucial for the stability analysis and sensitivity design. Finally, along with all parameters and signals in the proposed control law specified, we analyze the closed-loop stability and output tracking performance.

Key design equation: To proceed, we first review a fundamental design equation in the classical PPC method. The key equation is essential for the quantized control law design.

Choose a monic stable polynomial $A^*(s)$ of degree $2n + n_q - 1$, where n_q is the degree of the polynomial $Q_m(s)$

in (2). Under Assumption (A1), we can solve the following Diophantine equation

$$C(s)P(s)Q_m(s) + D(s)Z(s) = A^*(s), \tag{3}$$

with respect to $C(s)$ and $D(s)$ to find a solution of the form

$$C(s) = s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0, \tag{4}$$
$$D(s) = d_{n_q+n-1}s^{n_q+n-1} + \cdots + d_1 s + d_0. \tag{5}$$

Under Assumption (A1), the solution of the design equation (3) is unique for any $A^*(s)$ of degree $2n + n_q - 1$ (the proof can be seen in [20]).

Quantized-output feedback PPC law structure: To proceed, we give a monic stable polynomial of degree $n_q + n - 1$:

$$\Lambda_c(s) = s^{n_q+n-1} + \lambda^c_{n_q+n-2}s^{n_q+n-2} + \cdots + \lambda^c_1 s + \lambda^c_0.$$

Recalling $Q_m(s)$ below (2), we see that the coefficients q_i, $i = 0, 1, \ldots, n_q - 1$, and λ^c_j, $j = 0, 1, \ldots, n_q + n - 2$, are all known. With these parameters, motivated by [20], we design the quantized output feedback PPC law as

$$u(t) = \theta_1^{*T}\omega_1(t) + \theta_2^{*T}\omega_2(t) + \theta_3^*\omega_3(t), \tag{6}$$

where $\omega_1(t) \in \mathbb{R}^{n_q+n-1}$, $\omega_2(t) \in \mathbb{R}^{n_q+n-1}$ and $\omega_3(t) \in \mathbb{R}$ are of the form

$$\omega_1(t) = \frac{a_1(s)}{\Lambda_c(s)}[u](t),$$
$$\omega_2(t) = \frac{a_1(s)}{\Lambda_c(s)}\big[y^* - q(y, \Delta)\Delta\big](t),$$
$$\omega_3(t) = y^*(t) - q(y(t), \Delta(t))\Delta(t),$$

with $a_1(s) = [1, s, \ldots, s^{n_q+n-2}]^T$. Moreover, the parameters $\theta_1^* \in \mathbb{R}^{n_q+n-1}$, $\theta_2^* \in \mathbb{R}^{n_q+n-1}$ and $\theta_3^* \in \mathbb{R}$ are of the form

$$\theta_1^* = \begin{bmatrix} \lambda^c_0 \\ \lambda^c_1 \\ \vdots \\ \lambda^c_{n_q+n-2} \end{bmatrix} - \begin{bmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_0 \\ 1 & c_{n-2} & \cdots & c_1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_{n-1} \\ 0 & 0 & \cdots & c_{n-2} \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n_q-1} \\ 1 \end{bmatrix},$$

$$\theta_2^* = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n_q+n-2} \end{bmatrix} - d_{n_q+n-1}\begin{bmatrix} \lambda^c_0 \\ \lambda^c_1 \\ \vdots \\ \lambda^c_{n_q+n-2} \end{bmatrix},$$

$$\theta_3^* = d_{n_q+n-1}.$$

Note that $\omega_1(t)$ is the output of a system with $\frac{a_1(s)}{\Lambda_c(s)}$ as the transfer function and $u(t)$ as the input, and $\omega_2(t)$ is the output of a system with $\frac{a_1(s)}{\Lambda_c(s)}$ as the transfer function and $y^* - q(y, \Delta)\Delta$ as the input. Thus, these signals can be obtained.

One can verify that, with some algebraic manipulations, the control law (6) can be converted to the following form

$$u(t) = (\Lambda_c(s) - C(s)Q_m(s))\frac{1}{\Lambda_c(s)}[u](t) + D(s)\frac{1}{\Lambda_c(s)}\big[y^* - q(y, \Delta)\Delta\big](t), \tag{7}$$

where $C(s)$ and $D(s)$ are given in (4) and (5), respectively. The form (7) will be used for the stability analysis.

Tracking error equation: Define the tracking error and the quantized error as

$$e(t) = y(t) - y^*(t),$$
$$e_q(y(t), \Delta(t)) = \Delta(t)q(y(t), \Delta(t)) - y(t),$$

respectively. Now, we give the following lemma.

Lemma 1: Under Assumption (A1), the quantized-output feedback PPC law (6), applied to the system (1), ensures

$$e(t) = \frac{Z(s)D(s)}{A^*(s)}[e_q(y, \Delta)](t) + \epsilon(t), \tag{8}$$

for some exponentially decaying signal $\epsilon(t)$.

The proof of this lemma is given in the Appendix. The equation (8) implies that the absolute value of the tracking error, denoted as $|y(t) - y^*(t)|$ converges to zero exponentially if using exact output feedback, i.e., $e_q(y(t), \Delta(t)) = 0$. However, for the finite quantized-output feedback case, $e(t)$ is no longer exponentially decaying and may be unbounded due to the saturated limitation of feedback.

Some technical lemmas: To proceed, we give the following lemmas. These lemmas will specify some input-to-output relationships which are crucial for the stability analysis. Consider a system described by

$$Y(t) = H(s)[U](t), \; t \geq 0, \tag{9}$$

where $U(t)$ and $Y(t)$ are the input and the output, respectively, and $H(s)$ denotes the transfer function. Let

$$\alpha \triangleq \max_i\{\text{real parts of } \lambda_i(A^*(s))\}, \tag{10}$$

where $\lambda_i(A^*(s))$, $i = 1, 2, \ldots, 2n + n_q - 1$, denote the zeros of $A^*(s)$. Choose a constant parameter $\sigma > 0$ such that

$$\sigma + \alpha < 0. \tag{11}$$

Define

$$F_m(s) \triangleq \frac{Z(s)D(s)(s+\sigma)}{A^*(s)}, f_m(t) \triangleq \mathcal{L}^{-1}(F_m(s-\sigma)). \tag{12}$$

Then, (8) can be re-written as

$$e(t) = F_m(s)\frac{1}{s+\sigma}[e_q(y, \Delta)](t) + \epsilon(t). \tag{13}$$

Lemma 2 [20]: Consider the system (9). If $\|U\|_1 < \infty$ and $\|H\|_1 < \infty$, then $\int_0^t |Y(\tau)|d\tau \leq \|H\|_1 \int_0^t |U(\tau)|d\tau$.

Lemma 2 specifies an integral relationship between the input and the output, which will be used to prove Lemma 3.

Lemma 3: Consider the system (9). Let $h_\sigma(t) = \mathcal{L}^{-1}(sH(s-\sigma))$. If $\|h_\sigma\|_1 < \infty$, then

$$|Y(t)| \leq \|h_\sigma\|_1 \int_0^t \exp(-\sigma(t-\tau))|U(\tau)|d\tau.$$

The proof of this lemma is given in the Appendix. Lemma 3 gives the relationship between the output at any moment $Y(t)$ and an integral of the input $U(t)$, which will play an essential role to derive Lemma 4.

Lemma 4: The quantized-output feedback PPC law (7), applied to the system (1), ensures that

$$|e(t)| \leq \|f_m\|_1 \int_0^t \left|e_q(y(t), \Delta(t))\right| \exp(-\sigma(t-\tau))d\tau + |\epsilon(t)|, \quad (14)$$

for some exponentially decaying signal $\epsilon(t)$.

Proof: From (12), we have $F_m(s-\sigma) = \frac{Z(s-\sigma)D(s-\sigma)s}{A^*(s-\sigma)}$. Since $\sigma + \alpha < 0$, then $A^*(s-\sigma)$ is a stable polynomial. Thus, it follows from (12) that $\|f_m\|_1 < \infty$. By Lemma 3, (12) and (13), we can obtain that $|\frac{Z(s)D(s)}{A^*(s)}[e_q(y, \Delta)](t)| \leq \|f_m\|_1 \int_0^t |e_q(y(t), \Delta(t))| \exp(-\sigma(t-\tau))d\tau$. Then, by triangular inequality, the proof is completed. ■

Lemma 4 specifies a relationship between e and e_q. This inspires us to design the control law as follows and contributes to the establishment of Theorem 1.

Specification of the control law: To proceed, let

$$\beta \triangleq \max_i\{\text{real parts of } \lambda_i(P(s))\},$$
$$c \triangleq \frac{1}{2}\|f_m\|_1, \ \delta_1(t) \triangleq c_1 \exp(\mu t), \ c_1 > 0, \quad (15)$$

where $\lambda_i(P(s))$, $i = 1, 2, \ldots, n$, denote the zeros of $P(s)$ on the complex plane, $c_1 > 0$ is a chosen constant and μ satisfies $\alpha < \mu < 0$ with α defined in (10). Besides, we use d to represent an upper bound of $|y^*(t)|$ such that $|y^*(t)| \leq d, \ \forall t \geq t_0$. Then, we choose a constant λ such that

$$\max\{\beta, 0\} < \lambda \leq 2\sigma, \quad (16)$$

with σ defined in (11). If M satisfies

$$M > \frac{c}{\sigma} + d + c_1 + \frac{1}{2}, \quad (17)$$

then the quantized version PPC law (6) can be specified as

$$u(t) = \begin{cases} 0, & t \in [t_0, t_1), \\ \theta_1^{*T}\omega_1(t) + \theta_2^{*T}\omega_2(t) + \theta_3^*\omega_3(t), & t \in [t_1, \infty), \end{cases} \quad (18)$$

where

$$\Delta(t) = \begin{cases} \exp(\lambda(t-t_0)), & t \in [t_0, t_2), \\ \exp(\lambda(t_2-t_0)), & t \in [t_2, t_3), \\ \gamma\exp(\lambda(t_2-t_0)), & t \in [t_3, \infty), \end{cases} \quad (19)$$

with γ being a designed constant such that

$$\frac{1}{M-\frac{1}{2}}\left(\frac{c\exp(-\sigma t_2)}{\sigma} + \frac{d+\delta_1(t_2)}{\exp(\lambda(t_2-t_0))}\right) < \gamma < 1. \quad (20)$$

Moreover, $t_i, \ i = 1, 2, 3$, in (19) are defined as

$$t_1 \triangleq \inf\{t > t_0 : |q(y(t), \Delta(t))| \leq M - 1\},$$
$$t_2 \triangleq t_1 + \tau_0, \ t_3 \triangleq \inf\{t \geq t_2 : |q(y(t))| \leq g(t)\},$$

with τ_0 being a chosen time interval satisfying $\epsilon(t) \leq \delta_1(t), \ \forall t \geq t_2$, and

$$g(t) = c\frac{\exp(-\sigma t_2)}{\sigma} + \frac{|y^*(t)| + \delta_1(t)}{\Delta(t_2)} + \frac{1}{2}.$$

Remark 2: To help readers better understand the proposed PPC strategy, we give the following explanation. Since $y(t_0)$ cannot be measured, it cannot be sure whether $q(y(t_0), \Delta(t_0))$ is saturated or not. Thus, when $t \in [t_0, t_1)$, we set $u = 0$ but increase $\Delta(t)$ that grows faster than $|y(t)|$. The moment t_1 is the time when $q(y(t), \Delta(t))$ is for the first time measured to be unsaturated. From this moment, we change the PPC law from zero to non-zero as shown in (18). To ensure that the proposed PPC law (18) is independent of the decaying signal $\epsilon(t)$ associated with the system initial conditions, we find a moment t_2. The moment t_2 is the time when the designed decaying signal $\delta_1(t)$ is equal or larger than $\epsilon(t)$. By this way, the proposed PPC law (18) is surely independent of the system initial conditions. To achieve the tracking error as small as possible, we switch $\Delta(t)$ one more time at the moment t_3. In other words, at $t = t_3$, we change $\Delta(t)$ to $\gamma\Delta(t)$ with $\gamma \in (0, 1)$ under the condition that $q(y(t), \Delta(t)) \leq g(t)$. More details can be seen in the proof of Theorem 1.

Performance analysis: With the finite quantized-output feedback PPC law (18), we derive the following main result.

Theorem 1: Under Assumption (A1), the quantized-output feedback PPC law (18), applied to the system (1) with any unmeasurable $y(t_0) \in \mathbb{R}$, ensures that all closed-loop signals are bounded and the tracking error satisfies

$$|e(t)| \leq \frac{1}{\sigma}c\gamma\exp((\lambda-\sigma)t_2 - \lambda t_0) + \delta_1(t), \ \forall t \geq t_3, \quad (21)$$

where c and $\delta_1(t)$ are defined in (15); σ, λ and γ are designed constants satisfying (11), (16) and (20), respectively.

The proof of this theorem is given in the Appendix. Theorem 1 clarifies that the exponential convergence of $e(t)$ may not be achieved due to the saturation limitation of the quantizer, however, a bound on $e(t)$ can be specified. As shown in (21), the bound depends on two adjustable parameters σ and γ. As long as $\sigma + \alpha < 0$ and γ satisfies (20), one can choose a larger σ and a smaller γ to make the tracking error smaller. Next, we show that the tracking error can be made arbitrarily small if the saturation value M satisfies a specified condition.

Theorem 2: Under Assumption (A1), for any given value $\varepsilon > 0$, if M satisfies

$$M > \frac{c^2\exp(-\lambda t_0)}{\varepsilon\sigma^2} + \frac{c(d+c_1)}{\varepsilon\sigma} + \frac{1}{2}, \quad (22)$$

then the quantized-output feedback PPC law (18) with $\gamma = \frac{\varepsilon\sigma}{c\exp((\lambda-\sigma)t_2-\lambda t_0)}$, applied to the system (1) with any unmeasurable $y(t_0) \in \mathbb{R}$, ensures that all closed-loop signals are bounded and the tracking error satisfies

$$|e(t)| \leq \varepsilon + \delta_1(t), \ \forall t \geq t_3, \quad (23)$$

where c, c_1, d and $\delta_1(t)$ are defined in (15); σ and λ are designed constants satisfying (11) and (16), respectively.

The proof of this theorem is not difficult to perform, so we omit it for simplicity. Theorem 2 shows that $e(t)$ can converge to a residual set that can be made arbitrarily small if a specified condition on M is met. Note that it can be readily to verify whether the condition (22) is satisfied or not. If not, with (20), one can still choose σ and γ to make the tracking error small.

In this work, the tracking error does not converge to zero asymptotically. A key issue is that we do not fully utilize the zoom-in feature of the sensitivity $\Delta(t)$ in this paper. With adjusting the structures of the PPC law and the sensitivity, it is possible to achieve asymptotic output tracking, which is left as a future study.

IV. Simulation Study

This section demonstrates the design procedure and verifies the effectiveness of the proposed control method.

Simulation system: Considering the system (1), we set $P(s) = (s - \frac{1}{2})(s + 1), Z(s) = s - \frac{1}{3}$. We can see that

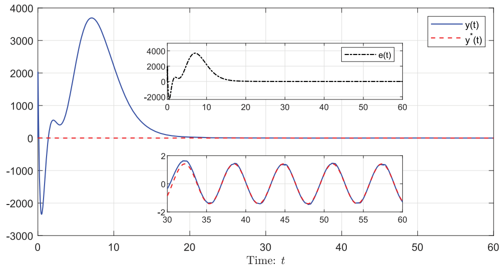

Fig. 1. Trajectories of $y(t)$, $y^*(t)$ and $e(t)$.

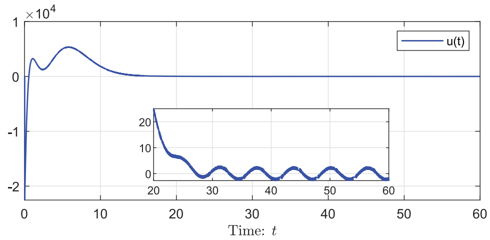

Fig. 2. Trajectory of $u(t)$.

this model is unstable with an unstable pole $s = \frac{1}{2}$ and is non-minimum phase with an unstable zero $s = \frac{1}{3}$. Since in our theoretical results the reference output needs to be bounded, we choose $y^*(t) = \sin(t) + \cos(t)$, and the corresponding Q_m can be $Q_m(s) = s^2+1$. What's more, we choose $sA^*(s) = (s+1)(s+\frac{2}{3})(s+\frac{3}{4})(s+\frac{4}{5})(s+\frac{5}{6})$.

Specification of θ_1^*, θ_2^*, θ_2^* and c: Set $C(s) = s+c_0$, $D(s) = d_3s^3+d_2s^2+d_1s+d_0$. By solving the equation $C(s)Q_m(s)P(s)+D(s)Z(s) = A^*(s)$, we obtain a unique solution: $C(s) = s - \frac{4841}{600}$, $D(s) = \frac{6971}{600}s^3 + \frac{223}{16}s^2 + \frac{8053}{600}s + \frac{4441}{400}$. Choose $\Lambda_c(s) = (s+1)^3$, and hence, from PPC law (6), the parameters are $\theta_1^* = [\frac{5441}{600}, 2, \frac{1782}{161}]^T$, $\theta_3^* = \frac{6971}{600}$ and $\theta_2^* = [-\frac{619}{1200}, -\frac{643}{30}, -\frac{8367}{400}]^T$. We now calculate c. Set $\sigma = 0.3$. Then the function f_m has three zeros $t_1^* = 1.1598$, $t_2^* = 2.4985$, and $t_3^* = 7.8563$, and therefore, $c = \frac{1}{2}\|f_m\| = \frac{1}{2}\int_0^\infty |f_m(t)|dt = 15.05$.

Simulation figures: From the settings above (17), we choose $\lambda = 0.6$, $\mu = -0.6$, $M = 2001$, $d = 2$, $c_0 = 1$, $c_1 = 1$. In particular, with $\Lambda_c(s) = (s+1)^3$, the signals $\omega_i(t)$, $i = 1, 2$ can be obtained. Consider the initial value $y(0) = 2050$. The parameters of t_1, t_2, t_3, τ_0 and γ are specified as $t_1 = 0.02s$, $t_2 = 3.02s$, $t_3 = 17.18s$, $\tau_0 = 3s$ and $\gamma = 0.0214$. Fig. 1 shows the response of the system output $y(t)$ *vs.* the reference output $y^*(t)$. It can be seen from Fig. 1 that the tracking performance is satisfactory when t is larger than about 35s. Fig. 2 shows the response of the quantized output feedback PPC law. These simulation figures are consistent with the theoretical results. The simulation study uses a numerical example. How to apply the proposed method for applications is currently under investigation.

V. Conclusion

In this paper, we have developed a PPC based solution to solve the finite quantized-output feedback output tracking control problem for a general class of continuous-time LTI systems. The control systems are allowed to have unstable poles and zeros. An analytical control law is constructed independent of the system's initial condition. We show that the proposed control law can achieve global closed-loop stability and bounded output tracking under the same condition as the classical PPC control method. It would be interesting to consider the following questions further: (i) Whether can the proposed control method be modified to achieve global asymptotic tracking? (ii) If the coefficients of $P(s)$ and $Z(s)$ are unknown, how to achieve adaptive control based on the proposed control method?

Appendix

Proof of Lemma 1: First, we show that if $y(t)$ can be measured exactly, i.e., $q(y(t), \Delta(t))\Delta(t) = y(t)$, the feedback control law (7) ensures that the tracking error $\epsilon(t)$ converges to zero exponentially as t goes to infinity. Operating both sides of (3) on $y(t)$, we have

$$C(s)Q_m(s)P(s)[y](t) + D(s)Z(s)[y](t) = A^*(s)[y](t). \quad (24)$$

Notice that the classical PPC law (7) can be written as

$$C(s)Q_m(s)[u](t) = D(s)\big[y^* - y\big](t), \quad (25)$$

which implies that

$$D(s)[y](t) = D(s)[y^*](t) - C(s)Q_m(s)[u](t). \quad (26)$$

Substituting (26) into (24), and noticing that $P(s)[y](t) = Z(s)[u](t)$, we have $A^*(s)[y](t) = Z(s)D(s)[y^*](t)$. By the boundedness of $y^*(t)$ and the stability of $A^*(s)$, we conclude $y(t)$ is bounded. Then operating both sides of (3) on $u(t)$, we have $C(s)Q_m(s)P(s)[u](t) + D(s)Z(s)[u](t) = A^*(s)[u](t)$. Substituting (25) into (24), and noticing that $P(s)[y](t) = Z(s)[u](t)$, we have $A^*(s)[u](t) = P(s)D(s)[y^*](t)$. Hence, $u(t)$ is bounded. Last, operating both sides of (3) on $y^*(t)$, we have $D(s)Z(s)[y^*](t) = A^*(s)[y^*](t)$, which shows that $A^*(s)[y - y^*](t) = 0$. By the stability of $A^*(s)$, we know that the tracking error $y(t) - y^*(t)$ converges to zero exponentially as t goes to infinity.

As for the quantized version, the quantized-output feedback PPC law structure (7) can be written as

$$\begin{aligned} u(t) &= (\Lambda_c(s) - C(s)Q_m(s))\frac{1}{\Lambda_c(s)}[u](t) \\ &\quad + \frac{D(s)}{\Lambda_c(s)}\big[y^* - y\big](t) + \frac{D(s)}{\Lambda_c(s)}[e_q(y, \Delta)](t), \end{aligned}$$

Ignoring the exponentially decaying term associated with the system initial conditions, it is equivalent to $C(s)Q_m(s)[u](t) - D(s)[y^* - y](t) = D(s)[e_q(y, \Delta)](t)$. Multiply both sides of this equation by $Z(s)$, we have $Z(s)C(s)Q_m(s)[u](t) - Z(s)D(s)[y^* - y](t) = Z(s)D(s)[e_q(y, \Delta)](t)$. With $Q_m(s)[y^*](t) = 0$ and $P(s)[y](t) = Z(s)[u](t)$, we add to both sides of this equation by $C(s)P(s)Q_m(s)[y^*](t)$, and hence, we can obtain $A^*(s)[e](t) = Z(s)D(s)[e_q(y, \Delta)](t)$. Since $A^*(s)$ is stable, there exists some exponentially decaying $\epsilon(t)$ such that (8) holds.

Proof of Lemma 3: First, we know that

$$\begin{aligned} |Y(t)| &= \left|H(s)(s+\sigma)\frac{1}{s+\sigma}[U](t)\right| \\ &\le \frac{1}{s+\sigma}(|H(s)(s+\sigma)[U]|)(t) \end{aligned}$$

$$= \exp(-\sigma t)\int_0^t |\exp(\sigma\tau)(H(s)(s+\sigma))[U](\tau)|d\tau. \quad (27)$$

Let $h(t)$ denote the inverse Laplace transform of $H(s)(s+\sigma)$. Since

$$\mathcal{L}(\exp(\sigma t)h(t)) = \int_0^\infty \exp(\sigma t)h(t)\exp(-st)dt$$
$$= \int_0^\infty h(t)\exp((\sigma - s)t)dt = H(s-\sigma)s,$$

that is, $\mathcal{L}^{-1}(H(s-\sigma)s) = \exp(\sigma t)h(t)$, then we have

$$\begin{aligned}&(H(s-\sigma)s)[(\exp(\sigma t)U(t)]\\ &= \int_0^t \mathcal{L}^{-1}(H(s-\sigma)s)[\exp(\sigma\tau)U(\tau)]d\tau\\ &= \exp(\sigma t)(H(s)(s+\sigma))[U](t). \quad (28)\end{aligned}$$

Then, $|Y(t)| \leq \exp(-\sigma t)\int_0^t |H(s-\sigma)s[\exp(\sigma\tau)U(\tau)]|d\tau$. By Lemma 2, it can be directly obtained that $|Y(t)| \leq \|h_\sigma(t)\|_1 \int_0^t \exp(-\sigma(t-\tau))U(\tau)d\tau$. ■

Proof of Theorem 1: For t_1, we know that when $t_0 < t < t_1$, the control is $u(t) = 0$. Since $\Delta(t) = c_1\exp(\lambda(t-t_0))$ and $\lambda > \max\{\beta, 0\}$, $\Delta(t)$ increases faster than $|y(t)|$. Then there exists a time instant t_1 such that $|y(t)| \leq (M-1/2)\Delta(t)$, for all $t \geq t_1$, that is, $|q(y(t), \Delta(t))| \leq M-1$.

For τ_0, since $\alpha < \mu < 0$, it follows from the definition of $\delta_1(t)$ in (15) that the exponentially decaying signal $|\epsilon(t)|$ decreases faster than $\delta_1(t)$. Therefore, there exists a time interval τ_0 such that $|\epsilon(t)| \leq \delta_1(t), \ \forall t \geq t_1 + \tau_0$.

Second, we analyze the tracking error. We now claim that when $t > t_2$ and $\Delta(t) = \Delta(t_2)$, the inequality $|y(t)| < \Delta(t_2)(M-\frac{1}{2})$ holds. If it is not true, according to the continuousness of $y(t)$ and the inequality that $|y(t_2)| < \Delta(t_2)(M-\frac{1}{2})$, there exists $t' > t_2$ such that $|y(t)| \leq \Delta(t_2)(M-\frac{1}{2})$, $\forall\, t_2 < t < t'$, and $|y(t')| = \Delta(t_2)(M-\frac{1}{2})$, i.e., $e_q(y(t), \Delta(t_2)) \leq \frac{1}{2}\Delta(t_2), \ \forall\, t_2 < t < t'$. Then from (14), we obtain

$$\begin{aligned}\left|y(t')\right| &\leq |y^*(t')| + |y(t') - y^*(t')| \leq \left|y^*(t')\right| + \left|\epsilon(t')\right|\\ &\quad + \|f_m\|_1 \int_{t_2}^{t'} \left|e_q(y(t), \Delta(t))\right| \exp(-\sigma(t'+t_2-\tau))d\tau\\ &\leq \left|y^*(t')\right| + c\Delta(t_2)\frac{\exp(-\sigma t_2) - \exp(\sigma t')}{\sigma} + \delta_1(t')\\ &\leq \Delta(t_2)\left(\frac{c\exp(-\sigma t_2)}{\sigma} + \frac{\left|y^*(t')\right| + \delta_1(t')}{\Delta(t_2)}\right)\\ &< \Delta(t_2)(M-\frac{1}{2}),\end{aligned}$$

which contradicts with $|y(t')| = \Delta(t_2)(M-\frac{1}{2})$. Therefore, we know that $y(t)$ will never saturate as long as $t \geq t_2$ with the sensitivity $\Delta(t_2)$. When $t > t_2$, with the sensitivity $\Delta(t_2)$, we obtain $e_q(y(t), \Delta(t)) \leq \frac{1}{2}\Delta(t_2)$. Then we get

$$\begin{aligned}&\left|y(t) - y^*(t)\right|\\ &\leq \frac{1}{2}\|f_m\|_1\Delta(t_2)\frac{\exp(-\sigma t_2) - \exp(-\sigma t)}{\sigma} + \delta_1(t)\\ &\leq c\Delta(t_2)\frac{\exp(-\sigma t_2)}{\sigma} + \delta_1(t).\end{aligned}$$

Hence, t_3 is well defined. When $t = t_3$, we change the control law with $\Delta(t) = \gamma\Delta(t_2)$. Then we obtain

$$\begin{aligned}|y(t)| &\leq \gamma\Delta(t_2)\left(\frac{c\exp(-\sigma t_2)}{\gamma\sigma} + \frac{|y^*(t)| + \delta_1(t)}{\gamma\Delta(t_2)}\right)\\ &\leq \Delta(t_3)(M-\frac{1}{2}), \ \forall t \geq t_3.\end{aligned}$$

Thus, $q(y(t), \Delta(t))$ will not saturate when $t \geq t_3$ with the sensitivity $\Delta(t_3)$. Therefore, the inequality (21) holds.

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